Worcester Polytechnic Institute
Department of Mathematical Sciences

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General Comprehensive Examination

Linear Algebra

Print Name: ____________________________ Sign: ____________________________

Problem 1: Let $\mathbf{P}_k(x)$ denote the vector space of polynomials with real coefficients of degree k or less in x. Consider the linear transformation $\mathsf{T} : \mathbf{P}_3(x) \to \mathbf{P}_1(x)$ given by second differentiation, i.e., by $\mathsf{T}(p) = p'' \in P_1(x)$ for $p \in P_3(x)$.

Find the matrix representation of T with respect to the bases $\{1+x, 1-x, x+x^2, x^2-x^3\}$ for $P_3(x)$ and $\{1, x\}$ for $P_1(x)$.

Problem 2: Let $A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & -4 & -4 \end{pmatrix}$.

(a) What is the rank of A?

(b) What is the determinant of A?

(c) Find the eigenvalues and eigenvectors of A.

(d) Find the characteristic polynomial of A.

(e) Find the transformation matrix M and its inverse such that $J = M^{-1}AM$ is the Jordan canonical form of A.

(f) Does it make a difference if you do your computations over the real numbers or over the complex numbers? Justify your answer.

Problem 3: This problem involves the matrix exponential $\exp(M)$ for a square matrix M.

(a) Compute $\exp(At)$ if $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

(b) Prove that, if $AB = BA$, then $\exp(A)\exp(B) = \exp(A+B)$.

(c) Prove that, if A is *skew-symmetric* (i.e., $A^\top = -A$) then $\exp(A)$ is an orthogonal matrix.

Problem 4: Let A be an $n \times n$ complex Hermitian matrix with largest eigenvalue λ_1. Let B be the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and first column of A. If μ_1 is the largest eigenvalue of B, prove that $\mu_1 \leq \lambda_1$.

Problem 5: Suppose that T is an $n \times n$ linear transformation over the field $\mathbb{Q}$ of rational numbers satisfying $\mathsf{T}^2 = \mathsf{T}^{-1} - \mathsf{T}$. Prove that $n \equiv 0 \mod 3$.

Problem 6: Let $V = C^\infty([0,1])$ be the real inner product space of infinitely differentiable functions on the interval $[0,1]$ with inner product

$$\langle f, g \rangle := \int_0^1 f(t)g(t)\,dt\ .$$

The differential operator $\mathsf{T} = \frac{d}{dt}$ is a linear operator on V. The Riesz Representation Theorem guarantees the existence and uniqueness of the *adjoint operator* T^* of T . Give the meaning of T^* and in the special case where $f(0) = f(1) = 0$, find a simple expression for the function $\mathsf{T}^* f$.